1 Tension versus three-point bending

In this question, we consider square cross-section bars of a brittle material with thickness b and length l tested in uniaxial tension and three-point bending (Figure 1), and whose tensile strength obeys Weibull statistics.

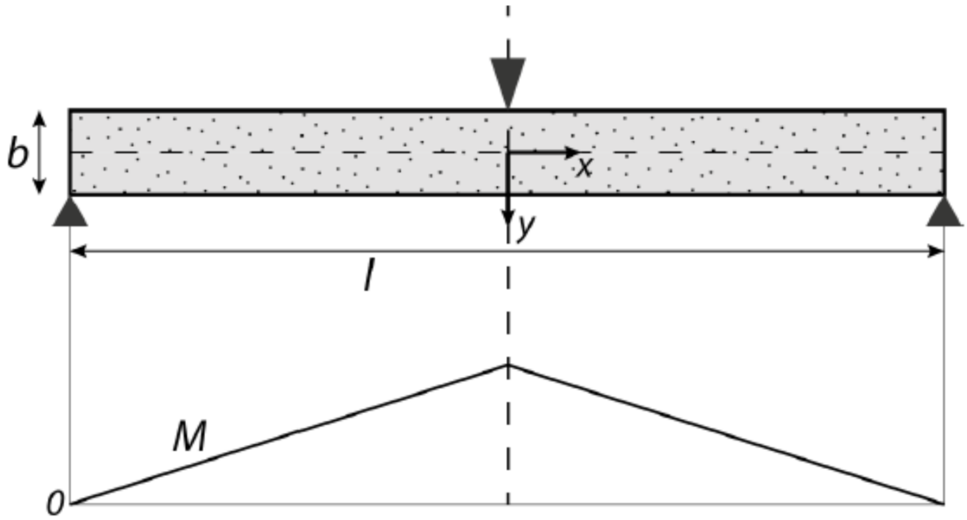

Figure 1: A three-point bend specimen along with a sketch of the variation of the bending moment M with distance x between the support points and the loading point.

Question 1

Write down the two-parameter Weibull distribution for failure in uniaxial tension. What is the corresponding expression for a solid with an arbitrary stress profile $\sigma(x, y)$, where x is in the tensile direction and y is perpendicular to the tensile direction? (You may assume σ not to vary in the third dimension).

Question 2

Show from simple beam analysis that in three-point bending:

$$\sigma(x, y) = \sigma_{max} \left(1 - \frac{2|x|}{l}\right) \frac{2y}{b}$$

where σ_{max} is the maximum stress in the specimen at any given load and you may assume from simple beam theory that $\sigma = M(x)y/I$, where y is the vertical distance from the neutral axis, I is the moment of inertia of the beam and M is the bending moment (for a square cross-section beam of side b, $I = b^4/12$, but you don't need this result to answer the question).

Question 3

Hence, derive the two-parameter Weibull distribution for failure in three-point bending in terms of the Weibull parameters obtained in uniaxial tension, and the expression for the correction factor k_m, namely $k_m = \frac{1}{2(m+1)^2}$.

2 Statistical failure of Si_3N_4 in four-point bending

Sintered silicon nitride is a very hard, high-performance ceramic used in car engines, bearings, and rocket engines, for example. It is therefore very much a structural material and must meet stringent performance requirements despite its intrinsically brittle nature.

A series of SI_3N_4 bars of constant rectangular square cross-section A were tested in four-point bending with an inner span length L_i equal to a third of the outer span length L_o. The strength data from 60 samples, are given at the end of these exercises, and in an Excel file on Moodle (if you need other formats, please ask), where they have already been ranked. You are asked to analyse these data using a two-parameter Weibull distribution.

ordered strength [Mpa]	ln(strength)	order	ordered strength [Mpa]	ln(strength)	order	ordered strength [Mpa]	ln(strength)	order
474	6.16120732	1	618.8	6.42778212	21	683.5	6.52722666	41
535.4	6.28301413	2	619.6	6.42907411	22	684	6.52795792	42
548.6	6.30736958	3	621	6.43133108	23	693.2	6.54131856	43
552.2	6.3139103	4	624.8	6.4374316	24	699.4	6.55022282	44
553	6.315358	5	626.2	6.43966981	25	699.7	6.55065167	45
561.6	6.33078985	6	626.3	6.43982949	26	699.7	6.55065167	46
561.7	6.3309679	7	631.3	6.44778119	27	706.9	6.56088921	47
578	6.35957387	8	632.7	6.44999638	28	707.2	6.56131351	48
580.6	6.36406205	9	636.3	6.45567015	29	708.8	6.5635734	49
583.7	6.36938715	10	640.7	6.46256133	30	710	6.56526497	50
584.7	6.3710989	11	642.2	6.46489978	31	712.7	6.56906057	51
588.1	6.376897	12	655.1	6.4847879	32	712.8	6.56920088	52
589.1	6.37859595	13	656.5	6.48692269	33	716.8	6.57479686	53
590.3	6.38063088	14	657.7	6.4887489	34	718.8	6.57758315	54
595.5	6.38940139	15	663.2	6.4970766	35	721.8	6.58174809	55
601.2	6.39892766	16	667.1	6.50293996	36	726	6.58755001	56
607.7	6.40968134	17	668	6.50428817	37	730.3	6.59345541	57
608.2	6.41050378	18	669	6.50578406	38	731.4	6.59496051	58
616.1	6.42340929	19	672	6.51025834	39	735.2	6.60014257	59
616.6	6.42422052	20	674.2	6.5135268	40	744.1	6.61217543	60

Figure 2: Ordered strength values of Si_3N_4 bars tested in four-point bending.

Question 1

Determine the Weibull modulus m. What is your opinion of the quality of the bars?

Question 2

Determine k_m for this geometry.

Question 3

Determine the Weibull scale parameter, σ_0, for a sample whose reference volume V_0 is the effective volume of the test bar (AL_i).

Question 4

What would the Weibull scale parameter for stress,σ_0, be if we had instead adopted as a reference volume a volume with the same length L_i but a cross-section equal to $A/16$?

Question 5

Your boss (who is fundamentally a nice person, but is under heavy pressure to cut costs), would like to guarantee to the customers that bars of length L_i and cross-section A will not fail below some critical stress σ_c. However, he/she can only afford to reject 30% of production. What do you do, and what should he/she tell the customers?

3 The fracture of solid fats

You do a series of tests on un-notched specimens of a solid fat-based confectionary formulation and find that its behaviour may be described by a two-parameter Weibull distribution with $m = 4$ and $\sigma_0 = 200$ kPa, owing to the presence of internal microcracks.

Question 1

Suppose that your material has a surface energy $\gamma = 3 \times 10^{-2}$ J/m^2. Estimate G_{IC} by assuming the material to behave as an ideal linear elastic solid up to fracture (cf. Griffith's model for glass).

Question 2

If the tensile modulus of the material is 200 MPa and its Poisson's ratio ν is 0.4, what is K_{IC}?

Question 3

Based on reasonable assumption that the microcracks that give rise to the Weibull distribution are penny-shaped, estimate the microcrack size that corresponds to the observed value of σ_0.

Question 4

Suppose that you can increase the tensile modulus of the specimen independently of all the other materials parameters (it might be enough to decrease the temperature to obtain roughly this effect). Do you think its strength will increase or decrease?